On the Douglas–Rachford operator in the (possibly) inconsistent case and related progress

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Dedicated to the memory of Jonathan Borwein

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Monotone operators

Throughout this talk

X is a real Hilbert space

with inner product $\langle \cdot, \cdot \rangle$, and induced norm $\|\cdot\|$.
Recall that an operator $A\colon X \rightrightarrows X$ is monotone if

$$(x, u), (y, v) \in \operatorname{gr} A \Rightarrow \langle x - y, u - v \rangle \geq 0.$$

Recall also that a monotone operator A is maximally monotone if A cannot be properly extended without destroying monotonicity.
In the following we assume that

A and B are maximally monotone operators on X.

The problem:
Find $x \in X$ such that

$$x \in \operatorname{zer}(A + B) := (A + B)^{-1}(0).$$

Connection to optimization

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- Choosing $A = \partial f$ and $B = \partial \iota_C = N_C$, the sum problem reduces to solving the constrained convex optimization:

$$\left.\begin{array}{l}\text{minimize } f(x)\\ \text{subject to } x \in C\end{array}\right\} \longrightarrow \text{find } x \in X \text{ such that } 0 \in (\partial f + N_C)x.$$

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- Choosing $A = \partial \iota_U = N_U$ and $B = \partial \iota_V = N_V$, the sum problem reduces to solving the convex feasibility problems:
  $$\text{find } x \text{ such } x \in U \cap V \longrightarrow \text{find } x \in X \text{ such that } 0 \in (N_U + N_V)x.$$

We shall use ι_U and N_U to denote the indicator function and the normal cone operator of a nonempty closed convex subset U of X.

Firmly nonexpansive operators and resolvents

Definition (resolvent and reflected resolvent)

The resolvent and the reflected resolvent of A are the operators

$$J_A := (\mathrm{Id} + A)^{-1}, \quad R_A := 2J_A - \mathrm{Id}.$$

Let $T : X \to X$. Then T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$.
T is firmly nonexpansive if $\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2$.

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Example

- Let $f : X \to \left]-\infty, +\infty\right]$ be proper lower semicontinuous convex function. Let $A := \partial f \Rightarrow J_A = (\mathrm{Id} + \partial f)^{-1} = \mathrm{Prox}_f$, where Prox_f is the Moreau prox operator of the function f.

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- Suppose that U is a nonempty closed convex subset of X. Let $A := N_U \Rightarrow J_A = (\mathrm{Id} + N_U)^{-1} = \mathrm{Prox}_{\iota_U} = P_U$.

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Fact

J_A is firmly nonexpansive and R_A is nonexpansive.

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The Douglas–Rachford splitting operator

The Douglas–Rachford splitting operator associated with the ordered pair (A, B) is

$$T := T_{A,B} := \mathrm{Id} - J_A + J_B R_A = \tfrac{1}{2}(\mathrm{Id} + R_B R_A).$$

The resolvent and the reflected resolvent of A are the operators $J_A := (\mathrm{Id} + A)^{-1}$ and $R_A := 2J_A - \mathrm{Id}$, respectively.
$\mathrm{zer}(A + B) = (A + B)^{-1}(0)$.

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- Thanks to Combettes, we know

$$J_A(\operatorname{Fix} T) = \operatorname{zer}(A + B).$$

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$$T^n x \xrightarrow[\textit{weakly}]{} \text{some point in } \operatorname{Fix} T \neq \operatorname{zer}(A+B).$$

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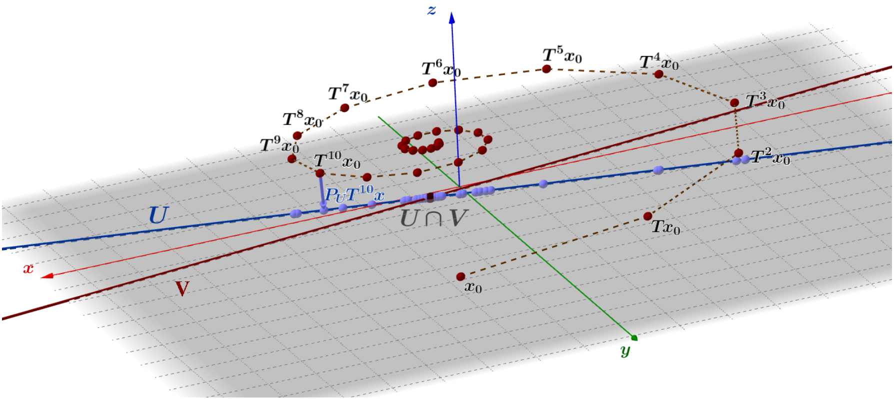
DR for two lines in $\mathbb{R}^3$

$$A = N_U,\ B = N_V \text{ and } T = \mathrm{Id} - P_U + P_V(2P_U - \mathrm{Id}).$$



U = the blue line,
V = the red line,
$(T^n x_0)_{n\in\mathbb{N}}$ = the red sequence,
$(P_U T^n x_0)_{n\in\mathbb{N}}$ = the blue sequence.

Motivation

Recall that when

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- Question: What happens when $\operatorname{zer}(A+B) = \varnothing$?

Inconsistent feasibility problem

Not every sum problem admits a solution:

- Suppose that U and V are nonempty closed convex subsets of X such that $U \cap V = \varnothing$.
- Set $A := N_U$ and $B := N_V$.
- Then $\operatorname{zer}(A+B) = (A+B)^{-1}(0) = U \cap V = \varnothing$.
- By an earlier fact[1] we have $\operatorname{zer}(A+B) = \varnothing \Leftrightarrow \operatorname{Fix} T = \varnothing$.

[1] Fact (Combettes (2004)): $J_A(\operatorname{Fix} T_{A,B}) = \operatorname{zer}(A+B)$.

The w-perturbed problem

Let $w \in X$ and $x \in X$. The corresponding inner and outer perturbations of A are

$$A_w x := A(x - w) \text{ and } {}_wAx := Ax - w.$$

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The w-perturbed problem associated with (A, B) is to find a point in the set of zeros

$$\begin{aligned} Z_w &:= \operatorname{zer}\left({}_w A, B_w\right) = \left({}_w A + B_w\right)^{-1}(0) \\ &= \left\{x \in X \;\middle|\; w \in Ax + B(x - w)\right\}. \end{aligned}$$

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Proposition

$$Z_w \neq \varnothing \Leftrightarrow w \in \operatorname{ran}(\operatorname{Id} - T).$$

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Corollary

$$\left\{w \in X \;\middle|\; Z_w \neq \varnothing\right\} = \operatorname{ran}(\mathrm{Id} - T).$$

The normal problem: Definition

The normal problem associated with (A, B) is to find a point in the set of zeros

$$Z_v := \operatorname{zer}\left({}_vA, B_v\right) = \left({}_vA + B_v\right)^{-1}(0) = \left\{x \in X \mid v \in Ax + B(x - v)\right\}.$$

where

$$v := v_{(A,B)} := P_{\overline{\operatorname{ran}}(\operatorname{Id} - T)}(0)$$

is the minimal displacement vector of (A, B) and the set of normal solutions is $Z_v = Z_{v_{(A,B)}}$.

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- $T_{({}_vA, B_v)} = T_{-v} = T(\cdot + v)$.
- If $(A, B) = (\partial\iota_U, \partial\iota_V) = (N_U, N_V)$ then

$$v = P_{\overline{U-V}}(0) \text{ and } Z_v = U \cap (v + V).$$

$T_v = T(\cdot - v)$.

Motivation

Recall that $U \cap V$ could be possibly empty. We recall also that

$$v := P_{\overline{U-V}}(0) = P_{\overline{\operatorname{ran}}(\operatorname{Id} - T)}(0).$$

In the following we assume that

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- $(\forall x \in X)\ P_U((T_{-v})^n x + v) \to$ a best approximation solution. (Fact)

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- $(\forall x \in X)\ P_U((T_{-v})^n x + v) \to$ a best approximation solution. (Fact)

Question: Can we come up with one algorithm that finds a best approximation solution and the gap vector (or even just a best approximation solution)?

$T_{-v} = T(\cdot + v)$.

Motivation

Fact (Pazy (1970))

Suppose that $T : X \to X$ *is nonexpansive such that* $\operatorname{Fix} T = \varnothing$. *Then* $(\forall x \in X)\ \|T^n x\| \to +\infty$.

Let $T : X \to X$. Then T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$.

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Fact (Pazy (1970))

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Fact (Bauschke-Combettes-Luke (2004))

Suppose that U *and* V *are nonempty closed convex subsets of* X *such that* $U \cap V = \varnothing$*. Then* $(\forall x \in X)$ *the shadow sequence* $(P_U T^n x)_{n\in\mathbb{N}}$ *is bounded and its weak cluster points lie in* $U \cap (v + V)$*, hence are best approximation solutions.*

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The case of infeasible affine subspaces: Example

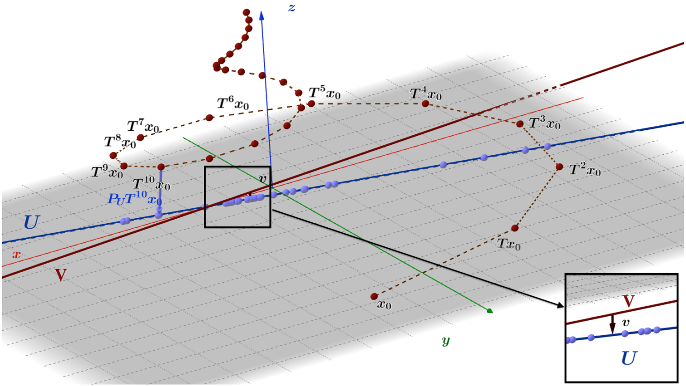

Figure: A GeoGebra snapshot. Two nonintersecting affine subspaces U (blue line) and V (red line) in $\mathbb{R}^3$. Shown are also the first few iterates of $(T^n x_0)_{n\in\mathbb{N}}$ (red points) and $(P_U T^n x_0)_{n\in\mathbb{N}}$ (blue points).

New useful identities

Let $(a, b, z) \in X^3$. Then

$$\|z\|^2 = \|z - a + b\|^2 + \|a - b\|^2 + 2\langle a, z - a\rangle + 2\langle b, 2a - z - b\rangle.$$

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Theorem

Let $x \in X$ and let $y \in X$. Then

$$\begin{aligned}\|x - y\|^2 = \|Tx - Ty\|^2 &+ \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \\ &+ 2\langle J_A x - J_A y, J_{A^{-1}} x - J_{A^{-1}} y\rangle \\ &+ 2\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y\rangle.\end{aligned}$$

Proof.

Apply the above identity with (a, b, z) replaced by $(J_A x - J_A y, J_B R_A x - J_B R_A y, x - y)$ and use that $T = \mathrm{Id} - J_A + J_B R_A$. □

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Corollary

Let $x \in X$ and let $y \in X$. Then the following hold:

$$(\mathrm{Id} - T)T^n x - (\mathrm{Id} - T)T^n y \to 0,$$
$$\langle J_A T^n x - J_A T^n y, J_{A^{-1}} T^n x - J_{A^{-1}} T^n y\rangle \to 0,$$
$$\langle J_B R_A T^n x - J_B R_A T^n y, J_{B^{-1}} R_A T^n x - J_{B^{-1}} R_A T^n y\rangle \to 0.$$

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This follows from the above theorem by telescoping. □

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- $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *that is Fejér monotone with respect to* E, *i.e.,*

$$(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|,$$

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$$(\forall e \in E) \ \ \langle u_n - e, u_n - x_n \rangle \to 0.$$

Then $(u_n)_{n\in\mathbb{N}}$ *converges weakly to some point in* E.

New Fejér monotonicity principle

Lemma

Suppose that

- E *is a nonempty closed convex subset of* X,
- $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *that is Fejér monotone with respect to* E, *i.e.,*

$$(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|,$$

- $(u_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* X *such that its weak cluster points lie in* E,
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$$(\forall e \in E) \ \langle u_n - e, u_n - x_n \rangle \to 0.$$

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Remark

$(x_n)_{n\in\mathbb{N}} = (u_n)_{n\in\mathbb{N}} \Rightarrow$ we recover the classical Fejér monotonicity principle!

New Fejér monotonicity principle: proof

- Step 1: $(\forall (e_1, e_2) \in E \times E) \quad \langle e_2 - e_1, u_n - x_n \rangle = \langle u_n - e_1, u_n - x_n \rangle - \langle u_n - e_2, u_n - x_n \rangle \to 0$.

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n \in \mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E) \ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n \in \mathbb{N}}$ converges weakly to some point in E.

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- Step 2: Obtain four subsequences $(x_{k_n})_{n\in\mathbb{N}}$, $(x_{l_n})_{n\in\mathbb{N}}$, $(u_{k_n})_{n\in\mathbb{N}}$ and $(u_{l_n})_{n\in\mathbb{N}}$ such that $x_{k_n} \rightharpoonup \bar{x}_1$, $x_{l_n} \rightharpoonup \bar{x}_2$, $u_{k_n} \rightharpoonup e_1$ and $u_{l_n} \rightharpoonup e_2$.

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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- We recall the following fact: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that is Fejér monotone with respect to a nonempty closed convex subset C of X. Let w_1 and w_2 be weak cluster points of $(x_n)_{n\in\mathbb{N}}$. Then $w_1 - w_2 \in (C - C)^\perp$.

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to* E, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \le \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E) \ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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- Step 3: Since $\{e_1, e_2\} \subseteq E$, applying the previous fact with (w_1, w_2, C) replaced by $(\bar{x}_1, \bar{x}_2, E)$ we conclude that $\langle e_2 - e_1, \bar{x}_2 - \bar{x}_1 \rangle = 0$.

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E) \ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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- Step 1: $(\forall (e_1, e_2) \in E \times E) \quad \langle e_2 - e_1, u_n - x_n \rangle = \langle u_n - e_1, u_n - x_n \rangle - \langle u_n - e_2, u_n - x_n \rangle \to 0$.
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Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to* E, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \le \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E) \ \langle u_n - e, x_n - u_n \rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

Application to the convex feasibility problem

Theorem

Suppose that U and V are nonempty closed convex subsets of X, and that $U \cap (v + V) \neq \varnothing$. Let $x \in X$. Then $(P_U T^n x)_{n \in \mathbb{N}}$ converges weakly to some point in $U \cap (v + V)$.

When $A = N_U$ and $B = N_V$ we have $T = T_{A,B} = \mathrm{Id} - P_U + P_V(2P_U - \mathrm{Id})$.

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Proof.

Let $y \in U \cap (v + V) \subseteq \mathrm{Fix}(v + T)$ (Fact).

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$$\langle P_U T^n x - y, T^n x + nv - P_U T^n x \rangle$$
$$= \langle P_U T^n x - y, T^n x - P_U T^n x - (y - nv - y) \rangle$$

□

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&= \langle P_U T^n x - P_U T^n y, (\operatorname{Id} - P_U) T^n x - (\operatorname{Id} - P_U) T^n y \rangle \\
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where the limit follows from an earlier result. □

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When $A = N_U$ and $B = N_V$ we have $T = T_{A,B} = \operatorname{Id} - P_U + P_V(2P_U - \operatorname{Id})$.

Application to the convex feasibility problem

Proof continued.

- Fact: $(T^n x + nv)_{n\in\mathbb{N}}$ is Fejér monotone w.r.t. $U \cap (v + V)$.

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n\rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

Application to the convex feasibility problem

Proof continued.

- Fact: $(T^n x + nv)_{n\in\mathbb{N}}$ is Fejér monotone w.r.t. $U \cap (v + V)$.
- Fact: $(P_U T^n x)_{n\in\mathbb{N}}$ is bounded and its weak cluster points lie in $U \cap (v + V)$.

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \leq \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n\rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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- Fact: $(T^n x + nv)_{n\in\mathbb{N}}$ is Fejér monotone w.r.t. $U \cap (v+V)$.
- Fact: $(P_U T^n x)_{n\in\mathbb{N}}$ is bounded and its weak cluster points lie in $U \cap (v+V)$.
- We just proved

$$\langle \underset{\substack{\parallel \\ u_n}}{P_U T^n x} - \underset{\substack{\cap \\ U\cap(v+V)}}{y}, \underset{\substack{\parallel \\ x_n}}{(T^n x + nv)} - \underset{\substack{\parallel \\ u_n}}{P_U T^n x} \rangle \to 0.$$

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \le \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n\rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

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- Fact: $(P_U T^n x)_{n\in\mathbb{N}}$ is bounded and its weak cluster points lie in $U \cap (v+V)$.
- We just proved

$$\langle \underbrace{P_U T^n x}_{= u_n} - \underbrace{y}_{\in U\cap(v+V)}, \underbrace{(T^n x + nv)}_{= x_n} - \underbrace{P_U T^n x}_{= u_n} \rangle \to 0.$$

- Apply the Fejér monotonicity lemma with $(E, (u_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}})$ replaced by $(U\cap(v+V), (P_U T^n x)_{n\in\mathbb{N}}, (T^n x + nv)_{n\in\mathbb{N}})$.

□

Lemma: Suppose that E is a nonempty closed convex subset of X, that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that is *Fejér monotone with respect to E*, i.e., $(\forall e \in E)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - e\| \le \|x_n - e\|$, that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that $(\forall e \in E)\ \langle u_n - e, x_n - u_n\rangle \to 0$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to some point in E.

Example

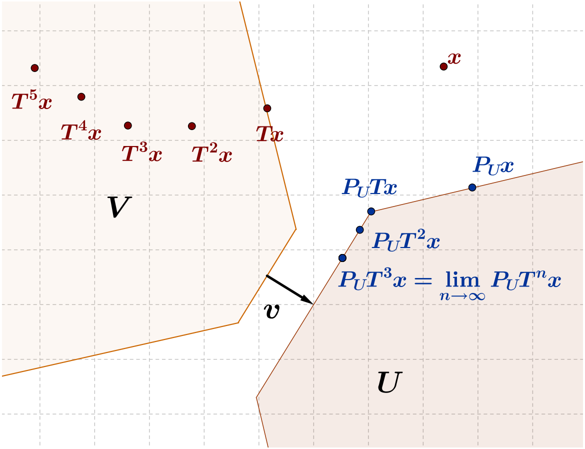

Figure: A `GeoGebra` snapshot. U and V are two nonintersecting sets in $\mathbb{R}^2$. Also, the first few iterates of the governing sequence $(T^n x)_{n\in\mathbb{N}}$ (red points) and the shadow sequence $(P_U T^n x)_{n\in\mathbb{N}}$ (blue points) are shown.

The Douglas–Rachford operator for two affine subspaces

In the following we set

$$T_{U,V} := T_{N_U, N_V},$$

where U and V are nonempty closed convex subsets of X.

Proposition

Suppose that U and V are affine subspaces of X. Set $A := N_U$, $B := N_V$ and $T := T_{U,V}$. Let $x \in X$. Then the following hold.

(i) $v \in (\operatorname{par} U)^{\perp} \cap (\operatorname{par} V)^{\perp}$.

Let U be an affine subspace of X. Then $\operatorname{par} U = U - U$.

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(ii) $(\forall \alpha \in \mathbb{R})\ P_U x = P_U(x + \alpha v)$.

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(i) $v \in (\operatorname{par} U)^{\perp} \cap (\operatorname{par} V)^{\perp}$.

(ii) $(\forall \alpha \in \mathbb{R})\ P_U x = P_U(x + \alpha v)$.

(iii) $(\forall n \in \mathbb{N})\ T^n x + nv = T^n_{U,v+V} x$.

Let U be an affine subspace of X. Then $\operatorname{par} U = U - U$.

Convergence of the shadows

Theorem

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(i) $(\forall n \in \mathbb{N})\ P_U T^n x = P_U T^n_{U,v+V} x$.

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Suppose that U and V are closed affine subspace of X such that $U \cap V \neq \varnothing$. Then $T^n x \to P_{\mathrm{Fix}\, T} x$, $P_U T^n x \to P_{U \cap V} x$, and $P_V T^n x \to P_{U \cap V} x$. If $\mathrm{par}\, U + \mathrm{par}\, V$ is closed then the convergence is linear with rate $c_F(\mathrm{par}\, U, \mathrm{par}\, V) < 1$.

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Step 3: Finally notice that $\mathrm{par}(v + V) = \mathrm{par}\, V$, hence if $\mathrm{par}\, U + \mathrm{par}\, V$ is closed then the convergence is linear with rate $c_F(\mathrm{par}\, U, \mathrm{par}\, V) < 1$, where c_F is the cosine of the Friedrichs angle between U and V.

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When one set is an affine subspace

Recall that

$$v := P_{\overline{U-V}}(0) \in \operatorname{ran}(\mathrm{Id} - T).$$

Theorem (convergence of DRA when U is a closed affine subspace)

Suppose that U is a closed affine subspace of X and that V is a nonempty closed convex subset of X. Let $x \in X$. Then

(i) *The shadow sequence $(P_U T^n x)_{n\in\mathbb{N}}$ converges weakly to some point in $U \cap (V + v)$.*

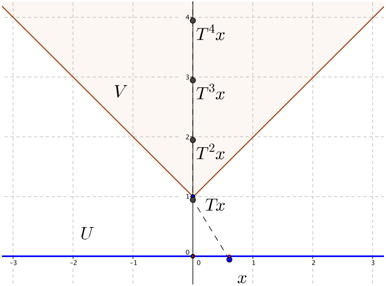
(ii) *No general conclusion can be drawn about the sequence $(P_V T^n x)_{n\in\mathbb{N}}$.*

Example

To prove: No general conclusion can be drawn about the sequence $(P_V T^n x)_{n\in\mathbb{N}}$. Recall that we proved the weak convergence of $(P_U T^n x)_{n\in\mathbb{N}}$ to a best approximation solution.

Example

Suppose that $X = \mathbb{R}^2$, that $U = \mathbb{R} \times \{0\}$ and that $V = \operatorname{epi}(|\cdot| + 1)$. Then $U \cap V = \varnothing$ and for the starting point $x \in [-1, 1] \times \{0\}$ we have $(\forall n \in \{1, 2, \ldots\})$ $T^n x = (0, n) \in V$ and therefore $\|P_V T^n x\| = \|T^n x\| = n \to +\infty$.



Application to the convex feasibility problems for more than two sets

Theorem

Suppose that $V_1, \ldots, V_M$ *are closed convex subsets of* X. *Set* $\mathbf{X} = X^M$, $\mathbf{U} = \big\{(x, \ldots, x) \in \mathbf{X} \;\big|\; x \in X\big\}$ *and* $\mathbf{V} = V_1 \times \cdots \times V_M$. *Let* $\mathbf{T} = \mathrm{Id} - P_{\mathbf{U}} + P_{\mathbf{V}}(2P_{\mathbf{U}} - \mathrm{Id})$, *let* $\mathbf{x} \in \mathbf{X}$ *and suppose that* $\mathbf{v} = (v_1, \ldots, v_M) := P_{\overline{\mathbf{U}-\mathbf{V}}}0 \in \mathbf{U} - \mathbf{V}$. *Then the shadow sequence* $(P_{\mathbf{U}}\mathbf{T}^n\mathbf{x})_{n\in\mathbb{N}}$ *converges to* $\bar{\mathbf{x}} = (\bar{x}, \ldots, \bar{x}) \in \mathbf{U} \cap (\mathbf{v} + \mathbf{V})$, *where* $\bar{x} \in \bigcap_{i=1}^{M}(v_i + V_i)$ *and* $\bar{x}$ *is a least-squares solution of*

$$\textit{find a minimizer of } \sum_{i=1}^{M} d_{V_i}^2.$$

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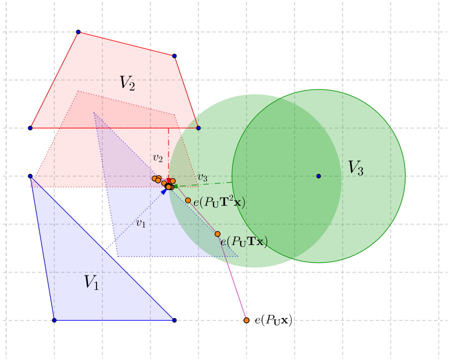

Figure: A GeoGebra snapshot. The DRA finds a point in the generalized intersection. Shown are the original sets as well the translated sets that forms this intersection.

And beyond feasibility!

Theorem

Suppose that

- *U is a closed affine subspace of X,*
- *$A = N_U$, that B is rectangular,*
- *$v = P_{\overline{\mathrm{ran}}(\mathrm{Id}-T)}0 \in \mathrm{ran}(\mathrm{Id}-T)$,*
- *$\mathrm{zer}({}_vA) \cap \mathrm{zer}(B_v) \neq \varnothing$ and*
- *all weak cluster points of $(J_A T^n x)_{n\in\mathbb{N}} = (P_U T^n x)_{n\in\mathbb{N}}$ lie in Z_v.*

Let $x \in X$. Then $(J_A T^n x)_{n\in\mathbb{N}} = (P_U T^n x)_{n\in\mathbb{N}}$ converges weakly to some point in Z_v.

Let $C\colon X \rightrightarrows X$. Then C rectangular (this is also known as paramonotone) if A is monotone and we have the implication

$$\left.\begin{array}{c}(x,u)\in \mathrm{gr}\,C\\(y,v)\in \mathrm{gr}\,C\\\langle x-y,u-v\rangle = 0\end{array}\right\} \quad\Rightarrow\quad \{(x,v),(y,u)\} \subseteq \mathrm{gr}\,C.$$

How far could the results be generalized?

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- One can show that $(\forall n \in \mathbb{N})\ T^n x = P_U x - nb$, hence $\|T^n x\| \to +\infty$.
- Consequently, $P_U T^n x = T^n x$, hence $\|P_U T^n x\| \to +\infty$ (unbounded!).

Convergence of shadows: Brief literature review

- Krasnosel'skiĭ–Mann (1950s)

$$T^n x \xrightarrow[\textit{weakly}]{} \text{some point in } \mathrm{Fix}\, T \neq \mathrm{zer}(A+B).$$

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Attouch–Théra duality and the Douglas–Rachford operator

The (Attouch–Théra) dual problem for the ordered pair (A, B) is to find a zero of $A^{-1} + B^{-\varovee}$, where $B := (-\operatorname{Id}) \circ B \circ (-\operatorname{Id})$. The primal (respectively dual) solutions are the solutions to the primal (respectively dual) problem given by

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Corollary

$$Z \times K = J_A(\operatorname{Fix} T) \times J_{A^{-1}}(\operatorname{Fix} T).$$

Proof.

Combine Combettes's result ($Z = J_A(\operatorname{Fix} T)$), applied to the primal and the dual problems, with Eckstein's above result.

□

Shadows' convergence: Useful identities

Recall that we proved earlier the useful identity:

$$\begin{aligned}\|x-y\|^2 = \|Tx-Ty\|^2 &+ \|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2\\ &+ 2\langle J_A x - J_A y, J_{A^{-1}}x - J_{A^{-1}}y\rangle\\ &+ 2\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y\rangle.\end{aligned}$$

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$$\begin{aligned}\|x-y\|^2 = \|Tx - Ty\|^2 &+ \|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2 \\ &+ 2\langle J_A x - J_A y, J_{A^{-1}}x - J_{A^{-1}}y\rangle \\ &+ 2\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y\rangle.\end{aligned}$$

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$$\begin{aligned}\|x-y\|^2 &= \|J_A x - J_A y + J_{A^{-1}}x - J_{A^{-1}}y\|^2 \\ &= \|J_A x - J_A y\|^2 + \|J_{A^{-1}}x - J_{A^{-1}}y\|^2 + 2\langle J_A x - J_A y, J_{A^{-1}}x - J_{A^{-1}}y\rangle.\end{aligned}$$

and

$$\begin{aligned}\|Tx - Ty\|^2 = \|J_A Tx - J_A Ty\|^2 &+ \|J_{A^{-1}}Tx - J_{A^{-1}}Ty\|^2 \\ &+ 2\langle J_A Tx - J_A Ty, J_{A^{-1}}Tx - J_{A^{-1}}Ty\rangle.\end{aligned}$$

Substituting in the first identity and simplifying yields:

$$\begin{aligned}&\|J_A x - J_A y\|^2 + \|J_{A^{-1}}x - J_{A^{-1}}y\|^2 - \|J_A Tx - J_A Ty\|^2 - \|J_{A^{-1}}Tx - J_{A^{-1}}Ty\|^2 \\ &= \|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2 + 2\langle J_A Tx - J_A Ty, J_{A^{-1}}Tx - J_{A^{-1}}Ty\rangle \\ &\quad + 2\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y\rangle.\end{aligned}$$

Shadows' convergence: Useful identities

We now have

$$\|J_A x - J_A y\|^2 + \|J_{A^{-1}} x - J_{A^{-1}} y\|^2 - \|J_A Tx - J_A Ty\|^2 - \|J_{A^{-1}} Tx - J_{A^{-1}} Ty\|^2$$
$$= \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 + \underbrace{2\langle J_A Tx - J_A Ty, J_{A^{-1}} Tx - J_{A^{-1}} Ty\rangle}_{\geq 0}$$
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Hence we conclude that

$$\|J_A Tx - J_A Ty\|^2 + \|J_{A^{-1}} Tx - J_{A^{-1}} Ty\|^2 \leq \|J_A x - J_A y\|^2 + \|J_{A^{-1}} x - J_{A^{-1}} y\|^2.$$

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We now have

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$$= \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 + \underbrace{2\langle J_A Tx - J_A Ty, J_{A^{-1}} Tx - J_{A^{-1}} Ty\rangle}_{\geq 0}$$
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Working in $X \times X$, we can just write

$$\|(J_A Tx, J_{A^{-1}} Tx) - (J_A Ty, J_{A^{-1}} Ty)\|^2 \leq \|(J_A x, J_{A^{-1}} x) - (J_A y, J_{A^{-1}} y)\|^2.$$

Shadows convergence: A simplified proof

Recall that the so-called Kuhn–Tucker set is defined by

$$\mathcal{S} := \mathcal{S}_{(A,B)} := \{(z,k) \in X \times X \mid -k \in Bz, k \in Az\} \subseteq Z \times K.$$

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Theorem

Suppose that $Z = \operatorname{zer}(A+B) \neq \varnothing$. Let $x \in X$ and let $(z,k) \in \mathcal{S}$. Then the following hold:

(i) *For every $n \in \mathbb{N}$, we have*
$\|(J_A T^{n+1}x, J_{A^{-1}} T^{n+1}x) - (z,k)\|^2 \leq \|(J_A T^n x, J_{A^{-1}} T^n x) - (z,k)\|^2$,
i..e., $(J_A T^n x, J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ is Fejér monotone with respect to $\mathcal{S}$.

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(i): We have $z + k \in \operatorname{Fix} T$ (details omitted).

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$(z,k) = (J_A(z+k), J_{A^{-1}}(z+k))$

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Proof.

(i): We have $z+k \in \operatorname{Fix} T$ (details omitted). Therefore $(\forall n \in \mathbb{N})$ $(z,k) = (J_A(z+k), J_{A^{-1}}(z+k)) = (J_A T^n(z+k), J_{A^{-1}} T^n(z+k))$. Apply $\|(J_A Tx, J_{A^{-1}} Tx) - (J_A Ty, J_{A^{-1}} Ty)\|^2 \leq \|(J_A x, J_{A^{-1}} x) - (J_A y, J_{A^{-1}} y)\|^2$ with (x,y) replaced by $(T^n x, z+k)$. (ii): We prove the weak cluster points of the bounded sequence $(J_A T^n x, J_{A^{-1}} T^n x)_{n\in\mathbb{N}}$ lie in $\mathcal{S}$ (details omitted). Now combine with (i) and use the classical Fejér monotonicity principle. □

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THANK YOU !!